Equal Sums of Quartics (In Context with the Richmond Equation) $\left(ax^4 + by^4 + cz^4 + dw^4 = 0\right)$

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Abstract: Consider the below mentioned Equation: $ax^4 + by^4 + cz^4 + dw^4 = 0 - - -$ [1] In section (1) we consider solution's with the condition on the coefficient's of equation[1]. Namely the product $(abcd)$ =square. In section [2] we consider the coefficients of Equation [1], with the product of coefficient's (abcd) not equal to a square. Historically Equation [1] has been studied by Ajai Choudhry, A. Bremner, M.Ulas [ref. 5] in 2014. Also Richmond [ref. 1 & 2] has done some ground work in 1944 & 1948. This paper has gone a step further, by finding many parametric solutions & new small numerical solutions by the use of unique Identities. The identities are unique, because they are of mixed powers(combination of quartic & quadratic variables) which are then converted to only degree four identities. As an added bonus in section [B], we came up with a few quartic $(4 - 1 - n)$ numerical solutions for $(n < 50)$ by elliptical mean's. A table of numerical solutions for the(4-1-n)Equation arrived at by brute force computer search is also given [ref # 7].

Keywords Quartic Equation, Diophantine Equation, Diagonal Quartic Surfaces, Number Theory

1. Section (1)

$$ax^4 + by^4 + cz^4 + dw^4 = 0 \tag{1}$$

[Condition:The product $(abcd)$ =square]

Existence of solution for diophantine equation
$$ax^4 + by^4 + cz^4 + dw^4 = 0$$
are known if abcd is square number.
Furthermore, condition of
$$a + b + c + d = 0$$
or having one known solution is necessary to have a rational solution.
Anyway, we show some parametric solutions of
$$ax^4 + by^4 + cz^4 + dw^4 = 0 \text{ with abcd is square number.}$$

Case (A)

We show Diophantine Equation,

$$2mnX^4 + (m^2 - n^2)Y^4 = 2mnZ^4 + (m^2 - n^2)W^4 \tag{1}$$

has infinitely many integer solutions. m, n are arbitrary.

We consider below Equation:
Where, in Equation (1), (a, b, c, d)=(p, q,-p,-q)

$$p(at + s)^4 + q(bt - 1)^4 = p(at - s)^2 + q(bt + 1)^2.$$
$$(8pa^3s - 8qb^3)t^3 + (8pas^3 - 8qb)t = 0 \tag{2}$$

To obtain a rational solution of Equation (2) for t, we have to find the rational solution of (3).

$$v^2 = -a^4s^4p^2 + (qb^3as^3 + a^3sqb)p - q^2b^4 \tag{3}$$

Let consider Equation (3) is a quadratic Equation for p.

$$Substitute\ p = \frac{b}{s}, q = a\ to\ equation\ (3), then$$

$$v^2 = -a^2b^2(s - 1)(s + 1)(a - b)(a + b).$$

$$Let\ b = m^2 + n^2, a = m^2 - n^2,$$

$$s = \frac{m^2 + n^2}{2(mn)},$$

$$then\ we\ obtain\ p = 2mn,$$

$$q = m^2 - n^2,$$

$$v = (m^2 + n^2)(m^2 - n^2)^2$$

$$t = \frac{n^2 - m^2}{4(m^2n^2)}.$$

Thus we obtain a parametric solution.

$$(p, q) = (2mn, m^2 - n^2).$$
$$X = m^4 - 2m^2n^2 + n^4 - 2m^3n - 2mn^3$$
$$Y = m^4 - n^4 + 4m^2n^2$$
$$Z = m^4 - 2m^2n^2 + n^4 + 2m^3n + 2mn^3$$
$$W = m^4 - n^4 - 4m^2n^2$$

Where, m,n are arbitrary.
Furthermore, we can find other parametric solution by using this method.
Hence by repeating this process, we can obtain infinitely many integer solutions for Equation (1).
Example:

$$(m, n) = (2,1):$$
$$4X^4 + 3Y^4 = 4Z^4 + 3W^4$$
$$(X, Y, Z, W) = (11, 31, 29, 1)$$
$$= \begin{pmatrix} 264009744251, 722953698209, \\ 676722505229, 70657231681 \end{pmatrix}$$
$$(m, n) = (3,2): 12X^4 + 5Y^4 = 12Z^4 + 5W^4$$
$$(m, n) = (4,1): 8X^4 + 15Y^4 = 8Z^4 + 15W^4$$

Case (B):

We show Diophantine Equation:

$$(m^4 + 1)X^4 + (m^4 + 1)Y^4 = 2Z^4 + 2W^4 \tag{1a}$$

$$has\ infinitely\ many\ integer\ solutions.$$

m is arbitrary.

Since in (1a) above, (a, b, c, d) = $[(m^4+1), (m^4+1), (2)(2)]$ the product (abcd) becomes a square=$[2(m^4+1)]$. If 'm' is written as (p/q) the above can be re-written as below:

$$(m^4+n^4)X^4 + (m^4+n^4)Y^4 = 2Z^4 + 2W^4.$$

m,n are arbitrary.

$$(m^4+1)X^4 + (m^4+1)Y^4 = 2Z^4 + 2W^4 \tag{1}$$

$$We\ use\ an\ identity\ p(t+1)^4 + p(t)^4$$

$$= (t^2+at+b)^2 + (ct^2+dt+e)^2, \tag{2}$$

$$with\ (a,b,c,e,p) = \left(d+2, \frac{d}{2}+1, 1+d, \frac{d}{2}, 1+d+\frac{d^2}{2}\right).$$

$$So,\ we\ look\ for\ the\ integer\ solutions\ \left\{ \begin{array}{l} Z^2 = t^2 + (d+2)t + \frac{d}{2} + 1, \\ W^2 = (1+d)t^2 + dt + \frac{d}{2} \end{array} \right\} \tag{3}$$

By parameterizing the first Equation and substituting the result to second Equation, then we obtain quartic Equation below to make (w= square).

Since, $(t, z) = \left(-\frac{1}{2}, \frac{1}{2}\right)$ we get $(z, w) = \left(\frac{1}{2}, \frac{m}{2}\right)$ so we parameterize for (t, z)

We get:

$$t = -\frac{(k^2+2k-2d-3)}{2(k^2-1)}$$

Hence we get:

$$v^2 = (d+1)k^4 + 4k^3 - 2k^2(d+1) - (12+8d^2+16d) + (9+17d+4d^3+12d^2) \tag{3a}$$

In-order to make the coefficient of $(k)^4$ a square we take $(d+1 = m^2)$

$$Hence\ let\ d = m^2 - 1,\ then\ substituting\ (d = m^2-1)\ in\ (3a)\ we\ get$$

$$v^2 = k^4m^2 + 4k^3 - 2k^2m^2 + (-8m^4-4)k + (4m^6+5m^2) \tag{4}$$

This quartic Equation (4) is bi-rationally equivalent to an elliptic curve below.

$$Y^2 - \frac{4YX}{m} + (8m+16m^5)Y = X^3 - \frac{2(m^4+2)X^2}{m^2} + (-20m^4-16m^8)X + 104m^6 + 32m^{10} + 80m^2$$

$$It\ has\ a\ point\ P(X,Y) = \left(\frac{2(m^4+2)}{m^2}, -\frac{16(-1+m^8)}{m^3}\right).$$

$$Hence\ we\ get\ 2P(X,Y) = \left(\frac{m^8+4m^4+1}{m^2}, -\frac{3m^8-m^4-3+m^{12}}{m^3}\right).$$

This point P is of infinite order, and the multiples "nP",

n = 2, 3, ...give infinitely many points.

This quartic Equation (4) above has infinitely many parametric solutions one of which is shown below.

$$n = 2:$$
$$X = m^4 + 4m^2 - 1$$
$$Y = m^4 - 4m^2 - 1$$
$$Z = 3m^4 + 1$$
$$W = (m^4+3)m$$
$$n=3$$
$$X = (m^{12} + 12m^{10} - 19m^8 + 40m^6 + 19m^4 + 12m^2 - 1)$$
$$Y = (m^{12} - 12m^{10} - 19m^8 - 40m^6 + 19m^4 - 12m^2 - 1)$$

$$Z = (5m^{12} - 27m^8 - 41m^4 - 1)$$
$$W = m(m^{12} + 41m^8 + 27m^4 - 5)$$

etc.

Hence we can obtain infinitely many integer solutions for Equation (1).
Example: m=2,

$$17X^4 + 17Y^4 = 2Z^4 + 2W^4:$$
$$[X, Y, Z, W] = [31, 1, 49, 38],$$
$$[14431, 15361, 12911, 30038], \dots etc.$$

Case (C):

$$(d^2 - 2dm^2 + 2m^4)X^4 + (d^2 - 2dm^2 + 2m^4)Y^4 = 2(-d + m^2)^2 Z^4 + 2W^4$$

We show Diophantine Equation,

$$(d^2 - 2dm^2 + 2m^4)X^4 + (d^2 - 2dm^2 + 2m^4)Y^4 = 2(-d + m^2)^2 Z^4 + 2W^4$$

has infinitely many integer solutions.
d, m are arbitrary.

$$(d^2 - 2dm^2 + 2m^4)X^4 + (d^2 - 2dm^2 + 2m^4)Y^4 = 2(-d + m^2)^2 Z^4 + 2W^4 \qquad (1)$$

We use an identity

$$p(t + 1)^4 + p(t)^4 = q(t^2 + at + b)^2 + (ct^2 + dt + e)^2, \qquad (2)$$

$$with \ (a, b, e, p, q) = \frac{d - 2c}{d - c}, \frac{(d - 2c)}{2(d - c)}, \frac{d}{2}, \frac{d^2 - 2dc + 2c^2}{2}, (c - d)^2$$

So, we look for the integer solutions

$$
\left\{
\begin{array}{c}
Z^2 = t^2 + \frac{(d-2c)t}{(d-c)} + \frac{d-2c}{2(d-c)}, \\[2mm]
W^2 = ct^2 + dt + \frac{d}{2}
\end{array}
\right\}
\qquad (3)
$$

By parameterizing the first Equation and substituting the result to second Equation, then we obtain quartic Equation (4) below.

$$Let \ c = m^2, then$$

$$u^2 = m^2(-d + m^2)^2 k^4 + (4m^6 - 12dm^4 + 12m^2d^2 - 4d^3)k^3 + (4dm^4 - 2m^2d^2 - 2m^6)k^2 + (-12m^6 + 20dm^4 - 12m^2d^2 + 4d^3)k + 5m^2d^2 + 9m^6 - 10dm^4 \qquad (4)$$

This quartic Equation (4) is bi-rationally equivalent to an elliptic curve below.

$$Y^2 - 4(-2d8m^2 + d^2 + m^4)\frac{XY}{m} + (24m^9 - 64dm^7 + 64m^5d^2 - 32m^3d^3 + 8md^4)Y$$
$$= X^3 - \frac{2(2d^4 - 8m^2d^3 + 13m^4d^2 - 10dm^6 + 3m^8)X^2}{m^2}$$
$$+ (80m^6d^3 - 20m^4d^4 - 136m^8d^2 + 112m^{10}d - 36m^{12})X + 80m^2d^8 - 640d^7m^4 + 2344d^6m^6$$
$$- 5104d^5m^8 + 7192d^4m^{10} - 6688d^3m^{12} + 3992d^2m^{14} - 1392dm^{16} + 216m^{18}$$

$$It \ has \ a \ point \ P(X, Y) = (\ 2(2d^4 - 8m^2d^3 + 13m^4d^2 - 10dm^6 + 3m^8)\frac{1}{m^2},$$

$$16(d^5 - 6m^2d^4 + 15m^4d^3 - 20m^6d^2 + 14m^8d - 4m^{10})\frac{d}{m^3})$$

$$Hence \ we \ get \ 2P(X, Y) = \left(\frac{d^4 - 4m^2d^3 + 10m^4d^2 - 12dm^6 + 6m^8}{m^2}, \frac{d(3d^5 - 18m^2d^4 + 46m^4d^3 - 64m^6d^2 + 48m^8d - 16m^{10})}{m^3} \right)$$

This point P is of infinite order, and the multiples nP, n = 2,3, ... give infinitely many points.

This quartic Equation has infinitely many parametric solutions below.

$$n = 2:$$

$$X = 4m^4 - 2dm^2 - d^2$$

$$Y = 4m^4 - 6dm^2 + d^2$$

$$Z = 4m^4 - 2dm^2 + d^2$$

$$W = m(4m^4 - 6dm^2 + 3d^2)$$

$$n = 3:$$

$$X = 64m^{12} - 224m^{10}d + 320m^8d^2 - 216m^6d^3 + 64m^4d^4 - 6d^5m^2 - d^6$$

$$Y = 64m^{12} - 160m^{10}d + 160m^8d^2 - 104m^6d^3 + 56m^4d^4 - 18d^5m^2 + d^6$$

$$Z = 64m^{12} - 224dm^{10} + 288d^2m^8 - 184d^3m^6 + 56d^4m^4 - 6d^5m^2 + d^6$$

$$W = m(64m^{12} - 160dm^{10} + 128d^2m^8 - 8d^3m^6 - 48d^4m^4 + 30d^5m^2 - 5d^6)$$

etc.

Hence we can obtain infinitely many integer solutions for Equation (1).
Example:

$$(d, m) = (2,2): \quad 10X^4 + 10Y^4 = 4Z^4 + W^4:$$

$$[X, Y, Z, W] = [11, 5, 13, 14], [563, 893, 277, 1646], \dots etc.$$

Case (D)

$$dm^2(-d - 2)X^4 + (-2m^2 + d + 2)m^2(d - 2m^2)Y^4 = m^4d(d - 2m^2)Z^4 + (d + 2)(2m^2 - d - 2)W^4$$

We show above Diophantine Equation has infinitely many integer solutions. d,m are arbitrary.

$$dm^2(-d - 2)X^4 + (-2m^2 + d + 2)m^2(d - 2m^2)Y^4 = m^4d(d - 2m^2)Z^4 + (d + 2)(2m^2 - d - 2)W^4 \quad (1)$$

We use an identity,

$$p(t + 1)^4 + q(t)^4 = r(t^2 + at + b)^2 + \frac{pq(ct^2 + dt + e)^2}{r} \quad (2)$$

$$with\ (a, b, e, p, q) = (\frac{d + 2}{c}, \frac{d + 2}{2c}, \frac{d}{2}, \frac{r(d + 2)}{c(-d + 2c)}, -\frac{(2c - d - 2)r}{dc})$$

$$So, we\ look\ for\ the\ integer\ solutions$$

$$\{Z^2 = (t^2 + \frac{(d + 2)t}{c} + \frac{d + 2}{2c}),$$

$$W^2 = \left(ct^2 + dt + \frac{d}{2}\right) \quad \}$$ (3)

By parameterizing the first Equation and substituting the result to second Equation, then we obtain quartic Equation below.

$$Let\ c = m^2, then$$

$$u^2 = m^4k^4 + (-4m^2d + 4m^4)k^3 + (4d^2 - 8m^2d + 6m^4 - 8m^2 + 8d)k^2 + (-16m^2 + 4m^4 - 4m^2d)k + 16 + m^4 - 8m^2 + 8d$$ (4)

This quartic Equation is bi-rationally equivalent to an elliptic curve below.

$$Y^2 + (-4d + 4m^2)YX + (-32m^4 + 8m^6 - 8m^4d)Y$$
$$= X^3 + (2m^4 - 8m^2 + 8d)X^2 + (-64m^4 - 4m^8 + 32m^6 - 32m^4d)X - 384m^8 - 8m^{12}$$
$$+ 96m^{10} - 96m^8d + 512m^6 + 512m^6d - 512m^4d - 256m^4d^2$$

$$It\ has\ a\ point\ P(X, Y) = (-2m^4 + 8m^2 - 8d, 64m^2d - 32d^2).$$

$$Hence\ we\ get\ 2P(X, Y) = (4 + 2m^4, 8m^4 + 8 + 16d - 16m^2).$$

This point P is of infinite order, and the multiples nP, n = 2,3, ...give infinitely many points.
This quartic Equation has infinitely many parametric solutions below.
n=2,

$$X = 3m^4 + (-2 - 2d)m^2 - 1$$
$$Y = m^4 + (-2 - 2d)m^2 + 1$$
$$Z = m^4 + 2m^2 - 2d - 3$$
$$W = m(m^4 - 2m^2 + 2d + 1)$$
$$n = 3:$$
$$X = 5m^{12} + (-10 - 10d)m^{10} + (8d - 9 + 4d^2)m^8 + (36 + 36d)m^6 + (-28d^2 - 56d - 29)m^4$$
$$+ (24d^2 + 8d^3 + 6 + 22d)m^2 + 1$$
$$Y = m^{12} + (-6 - 6d)m^{10} + (8d + 4d^2 + 15)m^8 + (-20 - 20d)m^6 + (40d + 20d^2 + 15)m^4$$
$$+ (-24d^2 - 8d^3 - 6 - 22d)m^2 + 1$$
$$Z = m^{12} + 6m^{10} + (-22d - 29)m^8 + (56d + 36 + 24d^2)m^6 + (-9 - 8d^3 - 28d^2 - 36d)m^4$$
$$+ (-8d - 10)m^2 + 4d^2 + 5 + 10d$$
$$W = m^{13} - 6m^{11} + (22d + 15)m^9 + (-40d - 20 - 24d^2)m^7 + (15 + 8d^3 + 20d^2 + 20d)m^5$$
$$+ (-8d - 6)m^3 + (4d^2 + 1 + 6d)m$$

etc.

Hence we can obtain infinitely many integer solutions for Equation (1).
Example:

$$(d, m) = (2,2):$$
$$-2X^4 + 6Y^4 = -12Z^4 + W^4$$

Or

$$12Z^4 + 6Y^4 = 2X^4 + W^4$$
$$[X, Y, Z, W] = [23, 7, 17, 26],$$
$$[647, 4183, 2993, 7274], \dots etc.$$

Equation of the type:

$$pa^m + qb^m + rc^n + sd^n = 0 \tag{1}$$

where: $(m, n) = (2,4)$

Note: We have attempted to parameterize a similar family of Equation related to Equation (1) at the top of the paper. Difference being Equation above is mixed powers of degree four & two. Others are free to attempt to make the two square components a fourth power. Thus making Equation (2) eligible for the Equation (1) group.

Case A:

$$pa^2 + qb^2 + rc^4 + sd^4 = 0 \tag{2}$$

Let,

$$(p, q, r, s) = (1,1, -1, -1)$$

Substitution above values in Equation (1) we get:

$$a^2 + b^2 = c^4 + d^4$$

Above is equivalent to:

$$(a + c^2)(a - c^2) = (d^2 + b)(d^2 - b)$$

Since, $(pq)(rs) = (ps)(rq)$, let:

$$(a + c^2) = pq$$
$$(a - c^2) = rs$$

$$(d^2 + b) = ps$$
$$(d^2 - b) = rq$$

Hence we get:

$$2c^2 = pq - rs$$
$$2d^2 = ps + qr$$
$$2a = pq + rs$$
$$2b = ps - qr$$
$$\text{Let, } q = 2(p + r) \quad \& \quad s = 2p$$

Therefore we get: $c = p$, &

$$d^2 = (p^2 + pr + r^2) \tag{2}$$

Above Equation (2) is parameterized as:

$$(p, r, d) = [(m^2 - n^2), (2mn + n^2), (m^2 + mn + n^2)]$$

Hence we get: $(c, d) = [(m^2 - n^2), (m^2 + mn + n^2)]$

Since, $2a = pq + rs$

we get, $a = (m^2 - n^2)(m^2 + 4mn + n^2)$

And as,

$$2b = ps - qr$$

we get, $b = (m^4 - 2m^3 n - 7m^2 n^2 - 2mn^3 + n^4)$

Summarizing, $a = (m^2 - n^2)(m^2 + 4mn + n^2)$

$$b = (m^4 - 2m^3 n - 7m^2 n^2 - 2mn^3 + n^4)$$
$$c = (m^2 - n^2)$$
$$d = (m^2 + mn + n^2)$$

for, $(m, n) = (3,2)$

$$(a, b, c, d) = (185,311,5,19)$$

& we get the numerical solution:

$$(5^4 + 19^4) = (185^2 + 311^2)$$

Additional Example:

Case B:

$$pa^m + qb^m + rc^n + sd^n = 0 \tag{1}$$
$$(p, q, r, s) = (5, -5, 1, -1)$$
$$(m, n) = (2,4)$$

Since we have,

$$5a^2 + c^4 = 5b^2 + d^4$$

Hence, $5(b^2 - a^2) = c^4 - d^4$

$$5(b + a)(b - a) = (c^2 + d^2)(c^2 - d^2),$$

Therefore we take,

$$c^2 - d^2 = (b + a)$$
$$c^2 + d^2 = 5(b - a)$$

Thus we have,

$$c^2 = (3b - 2a) \quad \& \quad d^2 = (2b - 3a)$$

Taking, $a = (10m^2 - 15n^2) \quad \& \quad b = (15m^2 - 10n^2)$

we get, $(c, d) = (5m, 5n)$

For, $(m, n) = (3,2)$ we get,

$$(a, b, c, d) = (30,95,15,10)$$

$$15^4 + 5(30)^2 = 10^4 + 5(95)^2$$

Note: Reader's may be interested in the table of result's below. These were arrived at by elliptical method. We have not attempted to parameterize the family of Equations below, but others can make an attempt. Since knowing the numerical solution is helpful towards parameterization.

Table 1. (below for numerical solutions):

Equation type: $ax^4 + by^4 = cz^4 + dw^4$ product (abcd) = □ (square), & Condition, (a+b)=(c+16d)	x	Y	Z	w
$x^4 + 20y^4 = 5z^4 + w^4$	19	5	13	2
$x^4 + 27y^4 = 12z^4 + w^4$	23	9	1	26
$x^4 + 49y^4 = 18z^4 + 2w^4$	73	23	39	22
$2x^4 + 18y^4 = 4z^4 + w^4$	13	3	11	2
$2x^4 + 28y^4 = 14z^4 + w^4$	17	7	1	22
$3x^4 + 25y^4 = 12z^4 + w^4$	9	131	157	92
$3x^4 + 49y^4 = 4z^4 + 3w^4$	1	31	41	58
$5x^4 + 20y^4 = 9z^4 + w^4$	17	7	15	10

2. Section (2)

$$ax^4 + by^4 + cz^4 + dw^4 = 0 \tag{1}$$

[Condition: Product of coefficient's (abcd) not equal to a square]

Case (A)

$$pX^4 + pY^4 + 2Z^4 = sW^4$$

Introduction:

Existence of solution for Diophantine Equation

$$ax^4 + by^4 + cz^4 + dw^4 = 0$$

are known if (abcd) is square number.

So, we are curious about whether above Equation has a solution or not if abcd is not a square number. In particular, when does this Equation have infinitely many integer solutions Since Richmond (ref # 2) conjectured that, if there is one rational solution there are infinite more.

We show Diophantine Equation: $px^4 + qy^4 + rz^4 = sw^4$

For (p,q,r,s)=(7,7,1,8)

$$7X^4 + 7Y^4 + Z^4 = 8W^4$$

has infinitely many integer solutions (numerical method).

Multiplying throughout by two the above Equation we get:

$$14X^4 + 14Y^4 + 2Z^4 = W^4$$

Which is a (4-1-30) quartic Equation:

$$7X^4 + 7Y^4 + Z^4 = 8W^4 \tag{1}$$

We use an identity

$$7(t + 4)^4 + 7(2t)^4 + (3t^2 + 24t - 16)^2 = 8\big(4(t^2 + t + 4)\big)^2 \tag{2}$$

$$So, we\ look\ for\ the\ integer\ solutions \begin{cases} Z^2 = 3t^2 + 24t - 16, \\ W^2 = 4t^2 + 4t + 16 \end{cases} \tag{3}$$

By parameterizing the second Equation and substituting the result to first Equation, then we obtain quartic Equation below.

$$u^2 = k^4 + 12k^3 - 26k^2 - 36k + 37 \tag{4}$$

This quartic Equation is bi-rationally equivalent to an elliptic curve below.

$$y^2 = x^3 + x^2 - 50x + 48.$$

$$Rank\ is\ 1\ and\ generator\ is\ [-2, -12].$$

Hence we can obtain infinitely many integer solutions for Equation (1).
Numerical example:

$$[X, Y, Z, W] = [48, 5, 37, 47], [11395, 83472, 87023, 104413]$$

From above we deduce:

$$W^4 = 14X^4 + 14Y^4 + 2Z^4 \tag{1}$$

$$Let: px^4 + qy^4 + rz^4 = w^4$$

$$For\ (p,q,r)=(14,14,2)$$

$$(X, Y, Z, W) = (48,5,37,94)$$

Let: n=(p+q+r)
Above Equation (1) is a (4-1-n) quartic Equation. For n=(14+14+2)=30 which we write as (4-1-n)=(4-1-30) Equation. There are more (4-1-n) numerical solution's as derived by elliptical method for (n < 50) & is shown below:

$$(94)^4 = 14(48)^4 + 14(5)^4 + 2(37)^4 \quad ----- \quad (4\text{-}1\text{-}30)\ eqn.$$

$$(7)^4 = 9(4)^4 + (1)^4 + 6(2)^4 \quad -- \quad ------- \quad (4\text{-}1\text{-}16)\ eqn.$$

$$(721)^4 = 36(200)^4 + (529)^4 + 3(460) \quad --- \quad (4\text{-}1\text{-}40)\ eqn.$$

$$(9)^4 = 25(4)^4 + (1)^4 + 10(2)^4 \quad ---- \quad ---(4\text{-}1\text{-}36)\ eqn.$$

$$(32)^4 = 12(17)^4 + 3(11)^4 + (7)^4 \quad -------- \quad (4\text{-}1\text{-}16)\ eqn.$$

<u>Note</u>: For more (4-1-n) Equation numerical solutions interested reader can see table (3) at the end of this paper, (ref # 12).

Parametric method in-lieu of the above elliptical numerical method (eqn. (4):
We show Diophantine Equation $pX^4 + pY^4 + 2Z^4 = sW^4$ has infinitely many integer solutions.

$$p = a^2 - 2, \quad s = a^2$$

$$pX^4 + pY^4 + 2Z^4 = sW^4 \tag{1}$$

We use an identity:

$$p(at + a)^4 + p(t)^4 + r(t^2 + a^2t + a^2)^2 = s(ct^2 + 2ct + a^2)^2 \tag{2}$$

$$with, \quad s = p + r, \quad r = \frac{2p}{a^2 - 2}, \quad c = a^2 - 1.$$

$$So, we\ look\ for\ the\ integer\ solutions \begin{cases} Z^2 = t^2 + a^2t + a^2, \\ W^2 = (a^2 - 1)t^2 + 2(a^2 - 1)t + a^2 \end{cases} \tag{3}$$

By parameterizing the second Equation and substituting the result to first Equation, then we obtain quartic Equation below.

$$u^2 = k^4 + (4a^2 - 6)k^2 - (+4a^4 + 8 - 12a^2)k + (-3 + 8a^2 - 5a^4 + a^6) \tag{4}$$

This quartic Equation is bi-rationally equivalent to an elliptic curve below.

$$Y^2 + (-8a^4 - 16 + 24a^2)Y$$
$$= X^3 + (4a^2 - 6)X^2 + (12 - 32a^2 + 20a^4 - 4a^6)X + (240a^2 - 248a^4 + 104a^6 - 16a^8 - 72).$$

The corresponding point is $P(X, Y) = (6 - 4a^2, 8a^4 - 24a^2 + 16)$.

Hence we get 2P(X,Y),

$$X = \frac{a^4 + 4a^2 + 12}{4},$$

$$Y = \frac{a^6 + 14a^4 - 132a^2 + 72}{8}$$

This point P is of infinite order, and the multiples mP, m = 2,3, ...give infinitely many points.
This quartic Equation has infinitely many parametric solutions below.

$$p = (-2 + a^2), \qquad s = a^2.$$

$$m = 2:$$

$$X = 8a(a^2 + 2)$$

$$Y = (a^2 - 4a - 2)(a^2 + 4a - 2)$$

$$Z = (-4 - 20a^2 + 3a^4)$$

$$W = (a^4 + 20a^2 - 12)$$

$$m = 3:$$

$$X = 8a(a^4 + 20a^2 - 12)^2(a^2 + 2)(-4 - 20a^2 + 3a^4)$$

$$Y = (a^4 + 20a^2 - 12)(a^2 - 4a - 2)(a^2 + 4a - 2) *$$

$$(a^4 - 8a^3 - 20a^2 - 16a + 4) * (a^4 + 8a^3 - 20a^2 + 16a + 4)$$

$$Z = (a^4 + 20a^2 - 12) *$$

$$(64 + 3008a^2 - 6096a^4 + 9120a^6 - 932a^8 - 228a^{10} + 5a^{12})$$

$$W = (a^4 + 20a^2 - 12) *$$

$$(a^{12} + 188a^{10} - 1524a^8 + 9120a^6 - 3728a^4 - 3648a^2 + 320)$$

etc.

Hence we can obtain infinitely many integer solutions for Equation (1).
Example:

$$a = 4: \quad 7X^4 + 7Y^4 + Z^4 = 8W^4$$

Numerical example:

$$We\ take\ (a, m) = (4,2)$$

$$[X, Y, Z, W] = [48, 5, 37, 47]$$

$$From\ above\ we\ deduce:$$

$$W^4 = 14X^4 + 14Y^4 + 2Z^4 \quad ----(4 - 1 - 30)$$

$$quartic\ numerical\ equation\ given\ below.$$

$$(94)^4 = 14(48)^4 + 14(5)^4 + 2(37)^4$$

Case (B):

$$pX^4 + pY^4 = 2Z^4 + W^4$$

We show Diophantine Equation $(pX^4 + pY^4 = 2Z^4 + W^4)$ has infinitely many integer solutions for $p = 1 + 2b^4$.
b is arbitrary.

$$p(X^4 + Y^4) = 2Z^4 + W^4. \tag{1}$$

We use an identity

$$p(t + 1)^4 + p(t)^4 = 2(t^2 + at + b^2)^2 + (ct^2 + dt + e)^2, \tag{2}$$

$$with\ p = 1 + 2b^4, a = 2b^2 + 1, c = -2b^2, d = 2 - 2b^2, e = 1.$$

So, we look for the integer solutions,

$$\{Z^2 = t^2 + (2b^2 + 1)t + b^2, W^2 = -2b^2t^2 + (2 - 2b^2)t + 1\} \tag{3}$$

By parameterizing the second Equation and substituting the result to first Equation, then we obtain quartic Equation below.

$$u^2 = k^4 + (-4b + 4b^3)k^3 + (-12b^4 + 2b^2)k^2 + (4b^3 + 4b + 16b^5)k - (4b^4 + 4b^2 + 8b^6 + 1) \tag{4}$$

This quartic Equation is bi-rationally equivalent to an elliptic curve below.

$$Y^2 + (-4b + 4b^3)YX + (8b^3 + 8b + 32b^5)Y = X^3 + (-4b^4 - 2b^2 - 4b^6)X^2 + (16b^4 + 16b^2 + 32b^6 + 4)X$$
$$-(192b^8 + 112b^6 + 192b^{10} + 48b^4 + 8b^2 + 128b^{12})$$

The corresponding point is $P(X,Y) = ((4b^4 + 2b^2 + 4b^6), (-24b^5 - 16b^9 - 8b))$.

$$Hence\ we\ get\ 2P(X,Y) = (\frac{4b^8 + 16b^6 - 12b^4 + 1}{4(b^2)},$$

$$(-)\frac{(52b^8 + 80b^6 + 18b^4 + 24b^{12} + 48b^{10} - 4b^2 - 1)}{8(b^3)})$$

This point P is of infinite order, and the multiples mP, m = 2,3, ...give infinitely many points.
This quartic Equation has infinitely many parametric solutions below.

$$p = 2b^4 + 1.$$
$$b\ is\ arbitrary.$$
$$m = 2:$$
$$X = (4b^8 + 20b^4 + 1)$$
$$Y = 8(2b^4 - 1)b^2$$
$$Z = b(-3 - 20b^4 + 4b^8)$$
$$W = (12b^8 + 20b^4 - 1)$$

$$m = 3:$$
$$X = (4b^8 + 20b^4 + 1)(16b^{16} + 416b^{12} + 152b^8 + 104b^4 + 1)$$
$$Y = 8(2b^4 - 1)(4b^8 - 20b^4 - 3)b^2(12b^8 + 20b^4 - 1)$$
$$Z = (64b^{24} - 3008b^{20} - 6096b^{16} - 9120b^{12} - 932b^8 + 228b^4 + 5)b$$
$$W = (1 - 188b^4 - 1524b^8 - 9120b^{12} - 3728b^{16} + 3648b^{20} + 320b^{24})$$

etc.

Hence we can obtain infinitely many integer solutions for Equation (1).
Example:

$$b = 1: \quad 3X^4 + 3Y^4 = 2Z^4 + W^4$$
$$b = 2: \quad 33X^4 + 33Y^4 = 2Z^4 + W^4$$

Case (C)

$$(d - 2)(d + 2)^2X^4 + (4d - 8)Y^4 + Z^4 = (4d + 8)W^4$$

We show Diophantine Equation, $(d - 2)(d + 2)^2X^4 + (4d - 8)Y^4 + Z^4 = (4d + 8)W^4$ has infinitely many integer solutions.
d is arbitrary.

$$(d - 2)(d + 2)^2X^4 + (4d - 8)Y^4 + Z^4 = (4d + 8)W^4 \tag{1}$$

$$We\ use\ an\ identity,\ p((t + 1))^4 + q(t)^4 = r(t^2 + at + a)^2 + s(ct^2 + dt + a)^2, \tag{2}$$

$$\text{with } (a,c,p,q,r) = \left(1 + \frac{d}{2}, \frac{d}{2}, \frac{s(-4+d^2)}{4}, \frac{s(-2+d)}{2+d}, -\frac{4s}{2+d}\right).$$

So, we look for the integer solutions

$$\begin{cases} Z^2 = t^2 + \left(1 + \frac{d}{2}\right)t + 1 + \frac{d}{2}, \\ W^2 = \frac{dt^2}{2} + dt + 1 + \frac{d}{2} \end{cases} \tag{3}$$

By parameterizing the second Equation and substituting the result to first Equation, then we obtain quartic Equation below.

$$u^2 = 16k^4 + (-32 + 32d)k^2 + (-16d^2 + 32d)k + (-8d^2 + 2d^3 + 16 + 8d) \tag{4}$$

This quartic Equation is birationally equivalent to an elliptic curve below.

$$Y^2 + (-128d^2 + 256d)Y = X^3 + (-32 + 32d)X^2 + (512d^2 - 128d^3 - 1024 - 512d)X$$
$$+(-32768d^2 + 20480d^3 + 32768 - 16384d - 4096d^4)$$

$$The\ corresponding\ point\ is\ P(X,Y) = (\ 32 - 32d, 128d^2 - 256d\ ).$$

$$Hence\ we\ get\ 2P(X,Y) = (\ d^2 + 12d + 68, -404d - 360 + 34d^2 + d^3).$$

This point P is of infinite order, and the multiples nP, n = 2,3, ...give infinitely many points.

This quartic Equation has infinitely many parametric solutions below.
For n=2

$$X = 16d + 96$$
$$Y = -d^2 + 36d + 60$$
$$Z = 2(3d^2 - 28d - 84)$$
$$W = d^2 + 44d + 36$$

For n=3,

$$X = 16(d^2 + 44d + 36)(d + 6)(3d^2 - 28d - 84)$$
$$Y = (d^2 - 36d - 60)(d^4 - 200d^3 - 616d^2 - 3360d - 5616)$$
$$Z = 2(316224 + 426816d + 215472d^2 + 25696d^3 - 7988d^4 - 396d^5 + 5d^6)$$
$$W = (46656 + 355392d + 262128d^2 + 39392d^3 - 2276d^4 + 388d^5 + d^6)$$

etc.

Hence we can obtain infinitely many integer solutions for Equation (1).

Example:

$$d = 1: \quad Z^4 = 12W^4 + 9X^4 + 4Y^4$$
$$d = 3: \quad 25X^4 + 4Y^4 + Z^4 = 20W^4$$

Re: d=1, Is equivalent to (4-1-25) Equation:

Case (D)

$$(m^2 + 1)^2(m^2 - 1)X^4 + (m^2 - 1)\,Y^4 + (m^2 + 1)Z^4 = 2m^2(W)^4$$

We show Diophantine Equation above has infinitely many integer solutions.
'm' is arbitrary.

$$(m^2 + 1)^2(m^2 - 1)X^4 + (m^2 - 1)\,Y^4 + (m^2 + 1)Z^4 = 2m^2W^4 \tag{1}$$

We use an identity:

$$p(t + 1)^4 + q(t)^4 + r(t^2 + at + b)^2 = s(ct^2 + dt + e)^2 \tag{2}$$

$$\text{with } (a,b,e,p,q,r) = (2, \frac{d}{-c+d}, d, -\frac{sd^2(d-2c)}{2c}, -\frac{sc(d-2c)}{2}, \frac{sd(d^2 + c^2 - 2cd)}{2c})$$

So, we look for the integer solutions

$$\{\ \ Z^2 = \frac{(t^2 + 2t + d)}{(-c+d)},\ \ W^2 = ct^2 + dt + d\ \} \tag{3}$$

By parameterizing the second Equation and substituting the result to first Equation, then we obtain quartic Equation below.

$$Let\ (c, d) = (m^2, m^2 + 1),\ then,$$

$$u^2 = m^2 k^4 + (-2m^4 + 4m^2)k^2 + (4m^3 - 4m)k + (m^6 + m^4 - 2m^2 + 1) \tag{4}$$

This quartic Equation is bi-rationally equivalent to an elliptic curve below.

$$Y^2 + (8m^4 - 8m^2)Y = X^3 + (-2m^4 + 4m^2)X^2 + (-4m^8 - 4m^6 + 8m^4 - 4m^2)X +$$
$$(8m^{12} - 8m^{10} - 32m^8 + 40m^6 - 16m^4)$$

The corresponding point is P(X,Y)=($2m^4 - 4m^2, -8m^4 + 8m^2$).

$$Hence\ we\ get\ 2P(X,Y) = \left(\frac{1}{8}(34m^4 + 12m^2 + 2), -\frac{1}{8}(45m^6 + 101m^4 - 17m^2 - 1)\right).$$

This point P is of infinite order, and the multiples nP, n = 2,3, ...give infinitely many points.

This quartic Equation has infinitely many parametric solutions below.
For n=2,

$$X = 8m(3m^2 + 1)$$
$$Y = 15m^4 + 18m^2 - 1$$
$$Z = 9m^4 + 22m^2 + 1$$
$$W = 21m^4 + 14m^2 - 3$$

For n=3,

$$X = 8(3m^2 + 1)(-3 + 14m^2 + 21m^4)m(1 + 22m^2 + 9m^4)$$
$$Y = (15m^4 + 18m^2 - 1)(351m^8 + 420m^6 + 154m^4 + 100m^2 - 1)$$
$$Z = 729m^{12} + 11106m^{10} + 16383m^8 + 4924m^6 - 569m^4 + 194m^2 + 1$$
$$W = 4941m^{12} + 13338m^{10} + 13467m^8 + 3212m^6 - 1997m^4 - 198m^2 + 5$$

etc.

Hence we can obtain infinitely many integer solutions for Equation (1).

Example:

$$m = 2:\ \ 75X^4 + 3Y^4 + 5Z^4 = 8W^4:$$
$$m = 3:\ \ 25X^4 + 4Y^4 + 5Z^4 = 9W^4$$

For m=2, multiplying throughout by two we get:
$(S^4 = 75P^4 + 3Q^4 + R^4)$ which is a (4-1-79) quartic Equation.

Example:

$$pa^m + qb^m + rc^n + sd^n = 0 \tag{1}$$

Note: We have attempted to parameterize a similar family of Equation related to Equation (1) at the top of the paper. Difference being Equation above is mixed powers of degree four & two. Others are free to attempt to make the two square components a fourth power. Thus making Equation (2) below eligible for the Equation (1) group above.

$$pa^m + qb^m + rc^n + sd^n = 0 \tag{2}$$

(pqrs) not a square,

$$(p, q, r, s) = (6, 1, -9, -1)$$
$$(m, n) = (2,4)$$

Hence Equation (1) becomes:

$$6a^2 + b^2 = 9c^4 + d^4 \tag{3}$$

Equation (3) has solution:

$$(a, b, c, d) = [(2t^2 + 6t + 3), (t^2 + t), ((t + 1), (t)]$$

For t=6, we get: $(a, b, c, d) = (7,6,42,111)$

$6(42)^2 + (111)^2 = 9(7)^4 + (6)^4$ After removing common factor's we get:

$$6(14)^2 + (37)^2 = 9(2)^4 + (7)^4$$

Also see table no. 4 at the end of this paper for results arrived at by brute force for Equation $(ax^4 + by^4 + cz^4 + dw^4 = 0)$. For (abcd) not equal to a square & k=(a+b+c+d) for (k < 30).

(See Table (2) below), for numerical solutions:

Note: Reader's may be interested in the table of result's below. These were arrived at by elliptical method. We have not attempted to parameterize the family of Equations below, but others can make an attempt. Since knowing the numerical solution is helpful towards parameterization.

Table (2) $aX^4 + bY^4 + cZ^4 + dW^4 = 0$	X	Y	Z	W
$23X^4 + 23Y^4 + 2Z^4 = 25W^4$	360	43	457	371
$X^4 + 36Y^4 + 27Z^4 = 4W^4$	51693	41613	38547	81294
$49X^4 + 49Y^4 + 8Z^4 = W^4$	3689	16892	17818	46819
$3X^4 + 75Y^4 + 5Z^4 = 8W^4$	22	15	17	28
$36X^4 + Y^4 + 3Z^4 = W^4$	200	529	460	721
$25X^4 + Y^4 + 10Z^4 = W^4$	4	1	2	9
$50X^4 + 2Y^4 + 5Z^4 = 2W^4$	72	1	12	161
$4X^4 + Y^4 + Z^4 = W^4$	72	49	84	113
$12X^4 + 12Y^4 = 8Z^4 + W^4$	9	17	19	14
$19X^4 + 19Y^4 = 2Z^4 + W^4$	8209	46228	81169	14397
$9X^4 + Y^4 + 2Z^4 = 3W^4$	40	47	37	57
$X^4 + 3Y^4 + Z^4 = W^4$	17	11	7	32

Equation: (table 3a)	X	Y	Z	
$5X^4 + 45Y^4 + Z^4 = 6W^4$	23	56	47	93
$2X^4 + 8Y^4 + Z^4 = 9W^4$	30	1	19	21
$31X^4 + 31Y^4 + Z^4 = 2W^4$	352	235	917	894
$25X^4 + 4Y^4 + Z^4 = 5W^4$	2	3	7	5
$79X^4 + 79Y^4 + 4Z^4 = 2W^4$	13	3	37	47
$57X^4 + 57Y^4 = 8Z^4 + 2W^4$	4	3	7	1
$18X^4 + 8Y^4 + Z^4 = 27W^4$	43	91	37	69
$73X^4 + 73Y^4 + Z^4 = 2W^4$	2361	5481	1758	13587
$75X^4 + 3Y^4 + 2Z^4 = 5W^4$	3	11	1	10
$33X^4 + 33Y^4 = 2Z^4 + W^4$	37	20	56	83
$3X^4 + 3Y^4 = 2Z^4 + W^4$	25	8	19	31
$9X^4 + 4Y^4 + 2Z^4 = 6W^4$	72	355	239	329
$25X^4 + 25Y^4 + 8Z^4 = 25W^4$	9	40	30	41
$X^4 + 9Y^4 + 6Z^4 = W^4$	1	4	2	7
$9X^4 + 36Y^4 + 4Z^4 = 9W^4$	1	12	6	17
$7X^4 + 7Y^4 + Z^4 = 8W^4$	48	5	47	37
$47X^4 + 47Y^4 + 2Z^4 = 49W^4$	952	475	2073	1123
$7X^4 + 7Y^4 + 2Z^4 = 9W^4$	264	95	59	249

[See Table (3a), below for (a,b,c,d) arbitary] $ax^4 + by^4 + cz^4 = dw^4$,

$$Equation: \ ax^4 + by^4 + cz^4 = w^4$$

$$(4 - 1 - n) \ numerical \ solution's: \ (\ 4 < n < 50\ )$$

$$where: \ n = (a + b + c)$$

Table (3b). $ax^4 + by^4 + cz^4 = w^4$

$n = (a + b + c)$	a	b	C	x	y	z	w
[5]	3	1	1	10	17	30	31
[6]	4	1	1	2	1	2	3
[7]	3	3	1	6	22	5	29
[8]	5	2	1	94	40	69	143
[10]	5	4	1	20	10	17	31
[11]	7	3	1	10	60	3	79
[12]	8	3	1	4	6	5	9
[13]	1	1	4	35	11	30	39
[14]	9	4	1	10	24	17	35
[15]	7	7	1	26	38	61	75
[16]	8	5	3	1	1	1	2
[17]	9	7	1	7	15	11	25
[18]	9	7	2	3	1	10	12
[19]	9	7	3	1	17	10	28
[20]	10	7	3	1	3	2	5
[21]	9	7	5	1	1	4	6
[22]	13	6	3	3	10	11	18
[23]	14	7	2	3	10	19	24
[24]	13	8	3	17	22	15	42
[25]	11	8	6	1	2	3	5
[26]	10	9	7	18	29	59	98
[27]	11	9	7	12	25	5	44
[28]	11	10	7	2	7	5	13
[29]	13	12	4	5	8	9	17
[30]	13	11	6	20	15	19	43
[31]	14	11	6	2	1	7	11
[32]	15	10	7	12	17	15	35
[33]	17	10	6	25	2	12	51
[34]	18	9	7	10	7	33	54
[35]	18	9	8	2	5	3	9
[36]	17	15	4	1	3	2	6
[37]	31	5	1	2	4	5	7
[38]	26	7	5	3	5	2	9
[39]	25	8	6	1	5	4	9
[40]	22	17	1	2	1	4	5
[41]	18	14	9	1	3	2	6
[42]	22	19	1	4	2	5	9
[43]	30	11	2	1	4	5	8
[43]	33	8	2	7	16	20	31
[44]	24	19	1	4	10	21	25
[45]	19	14	12	5	3	6	13
[46]	19	17	10	2	3	6	11
[47]	43	3	1	14	8	45	49
[48]	23	13	12	3	5	1	10
[49]	22	19	8	1	3	5	9

Table 4. (See below) $(ax^4 + by^4 + cz^4 + dw^4 = 0)$

Serial nos.	a	b	c	d	x	y	z	w
1	6	-3	-3	1	23	1	47	60
3	5	-2	-2	2	4	2	5	1
4	2	1	-1	2	2	3	5	4
5	4	2	-2	1	2	3	37	44
6	5	4	-4	1	20	17	31	40
7	7	-3	-3	6	6	5	7	1
8	5	2	-2	3	18	7	23	10
9	8	6	-6	1	10	5	11	8
10	7	2	-2	3	2	1	3	2
11	8	5	-5	3	10	9	13	10
12	7	5	-3	3	4	2	5	1
13	7	3	-3	6	3	13	30	25
14	11	2	-2	3	5	9	10	1
15	8	4	-4	7	26	7	31	8
16	10	7	-2	1	15	11	47	55
17	10	8	-8	7	4	3	5	4
18	13	3	-2	4	5	1	8	2
19	12	4	-2	5	1	5	6	2
20	8	7	-3	8	1	2	4	3
21	11	5	-2	7	5	9	12	4
22	12	6	-5	9	5	8	9	3
23	9	8	-2	8	2	3	10	7
24	9	8	-3	9	5	4	11	7
25	12	6	-3	10	4	1	6	3
26	13	7	-4	10	1	11	15	10
27	11	7	-1	10	2	5	13	7
28	15	1	-1	13	2	3	19	10
29	12	9	-3	11	5	7	10	3

coefficients (a,b,c,d) for Equation (1) a clear relationship has not emerged between the coefficient's (a,b,c,d). For example,in,some,cases,we,took:(a,b,c,d)=(23,23,2,25)=(1, 36,27,4)=(49,49,8,1)=(3,75,5,8).

The interested reader may try to find a pattern in the above (a,b,c,d), which could facilitate discovering more similar Identities of the type given for case (1) in section (B) shown below for example:
$$7(t + 4)^4 + 7(2t)^4 + (3t^2 + 24t − 16)^2 = 8(4t^2 + 4t + 16)^2.$$

Conclusion:
Re: Equation, $(ax^4 + by^4 + cz^4 + dw^4) = 0$ ----(1), while in section (A) we noticed, that when we imposed condition's on Equation (1) like (abcd) = □ & ((a+b)=(c+16d)), we got small numerical solutions for the variable's (X,Y,Z,W) [see table (1)]. But in section (B) in table (2a & 2b) when we used different numerical values of

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